$$B_f = [\, I_\mu : -A_{11}^T (A_{12}^T)^{-1} \,] \qquad (4\text{-}8)$$

where I_μ is the identity matrix of order $\mu = m - r$.

Exercise 4.3 Find the B_f matrix for the incidence matrix A found in Exercise 4.2.

Answer: Since the sum of the three rows in the matrix is zero, rank (A) = $r \leq 2$. We have a nonsingular submatrix of order 2. Thus $r = 2$ and $\mu = m - r = 4 - 2 = 2$. The matrix A shown in the answer of Exercise 4.2 is already partitioned in the form of (4-7), where

$$A_{11} = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \text{ and } A_{12} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}. \text{ We compute } A_{12}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$$

$$I_\mu = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } -A_{11}^T (A_{12}^T)^{-1} = -\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} = \begin{bmatrix} 2 & 1/2 \\ -1 & -1/2 \end{bmatrix}$$

Therefore, the matrix B_f can be found by (4-8):

$$B_f = \begin{bmatrix} 1 & 0 & : & 2 & 1/2 \\ 0 & 1 & : & -1 & -1/2 \end{bmatrix}$$

Note that $AB_f^T = 0$. That is, the vector space spanned by the row vectors of A is orthogonal to the vector space spanned by the row vectors of B_f. The matrix B_f corresponds to the fundamental circuit matrix [13] in the case of a marked graph. Now, the condition that ΔM is orthogonal to every solution for $Ay = 0$ is equivalent to the following condition,

$$B_f \Delta M = 0. \qquad (4\text{-}9)$$

Thus, if M_d is reachable from M_0, then the corresponding firing count vector x must exist and (4-9) must hold. Therefore, we have the following necessary condition for reachability in an unrestricted Petri net [198].